

1. Find the expression for the modified wave number in the following centered difference approximations to  $(\delta_x u)_j$  in terms of  $\Delta x$  and  $k$ . This is done just as done in class where we let  $u_j = e^{ikj\Delta x}$ . (Cast the results in terms of  $\sin(k\Delta x)$  and  $\cos(k\Delta x)$ ).

(a)  $(\delta_x u)_j = (u_{j+1} - u_{j-1})/(2\Delta x)$

We apply  $u_j = e^{ikj\Delta x}$  to both sides and get

$$(ik' e^{ikj\Delta x}) = e^{ikj\Delta x} (e^{+ik\Delta x} - e^{-ik\Delta x}) / (2\Delta x)$$

which give us

$$ik' = i \frac{\sin(k\Delta x)}{\Delta x}$$

(b)  $(\delta_x u)_j = (-u_{j+2} + 8u_{j+1} - 8u_{j-1} + u_{j-2})/(12\Delta x)$

We apply  $u_j = e^{ikj\Delta x}$  to both sides and get

$$(ik' e^{ikj\Delta x}) = e^{ikj\Delta x} (-e^{2ik\Delta x} + 8e^{+ik\Delta x} - 8e^{-ik\Delta x} + e^{-2ik\Delta x}) / (12\Delta x)$$

which give us

$$ik' = i \frac{\left(\frac{4}{3}\sin(k\Delta x) - \frac{1}{6}\sin(2k\Delta x)\right)}{\Delta x}$$

(c)  $\frac{1}{6}((\delta_x u)_{j+1} + 4(\delta_x u)_j + (\delta_x u)_{j-1}) = (u_{j+1} - u_{j-1})/(2\Delta x)$

We apply  $u_j = e^{ikj\Delta x}$  to both sides and get

$$\frac{1}{6}e^{ikj\Delta x} (ik' e^{+ik\Delta x} + 4ik' + ik' e^{-ik\Delta x}) = e^{ikj\Delta x} (e^{+ik\Delta x} - e^{-ik\Delta x}) / (2\Delta x)$$

which give us

$$\frac{1}{3}(2 + \cos(k\Delta x)) ik' = i \frac{\sin(k\Delta x)}{\Delta x}$$

or

$$ik' = i \frac{3\sin(k\Delta x)}{\Delta x(2 + \cos(k\Delta x))}$$

2. Find the expression for the modified wave number in the following one sided difference approximations to  $(\delta_x u)_j$  in terms of  $\Delta x$  and  $k$ . In this case there will be real and imaginary parts to the modified wave number. (Cast the results in terms of  $\sin(k\Delta x)$  and  $\cos(k\Delta x)$ ).

(a)  $(\delta_x u)_j = (u_j - u_{j-1})/\Delta x$  We apply  $u_j = e^{ikj\Delta x}$  to both sides and get

$$(ik' e^{ikj\Delta x}) = e^{ikj\Delta x} (1 - e^{-ik\Delta x}) / (2\Delta x)$$

which give us

$$ik' = \frac{1 - \cos(k\Delta x)}{\Delta x} + i \frac{\sin(k\Delta x)}{\Delta x}$$

(b)  $(\delta_x u)_j = (3u_j - 4u_{j-1} + u_{j-2})/(2\Delta x)$

We apply  $u_j = e^{ikj\Delta x}$  to both sides and get

$$(ik' e^{ikj\Delta x}) = e^{ikj\Delta x} (3 - 4e^{-ik\Delta x} + e^{-2ik\Delta x}) / (2\Delta x)$$

which give us

$$ik' = \frac{3 - 4\cos(k\Delta x) + \cos(2k\Delta x)}{2\Delta x} + i \frac{4\sin(k\Delta x) - \sin(2k\Delta x)}{2\Delta x}$$

$$(c) 2(\delta_x u)_j + (\delta_x u)_{j-1} = (u_{j+1} + 4u_j - 5u_{j-1}) / (2\Delta x)$$

We apply  $u_j = e^{ikj\Delta x}$  to both sides and get

$$e^{ikj\Delta x} (2ik' + ik'e^{-ik\Delta x}) = e^{ikj\Delta x} (e^{+ik\Delta x} + 4 - 5e^{-ik\Delta x}) / (2\Delta x)$$

which give us

$$(2 + \cos(k\Delta x) - i\sin(k\Delta x)) ik' = \frac{2 - 2\cos(k\Delta x) + 3i\sin(k\Delta x)}{\Delta x}$$

or

$$ik' = \frac{2 - 2\cos(k\Delta x) + 3i\sin(k\Delta x)}{(2 + \cos(k\Delta x) - i\sin(k\Delta x)) \Delta x}$$

You can avoid doing all the complex algebra and just evaluate these terms in complex arithmetic for the next problem.

3. For problems 1 and 2 plot the resulting expressions for the modified wave number against  $k$  for  $k = 1, 2, \dots, M$  with  $M = 51$  and  $\Delta x = 2\pi/M$ . (You can use Matlab or anything else you want.)

Here's a sample Matlab code for my plots.

```

clear;

M = input('Enter M = ');

dx = 2.*pi/M;

k = [1:M/2];

k_p1 = sin(k*dx)/dx;
k_p2 = 4./3.*sin(k*dx)/dx - 1./6.*sin(2.*k*dx)/dx;
k_p3 = 3.*sin(k*dx)/dx./(2.+cos(k*dx));

figure(1);
clf;

plot(k,k_p1,'w',k,k_p2,'w--',k,k_p3,'w:');
title('Modified Wave Number Prob 1 (a:solid), (b:dashed), (c:dotted)');

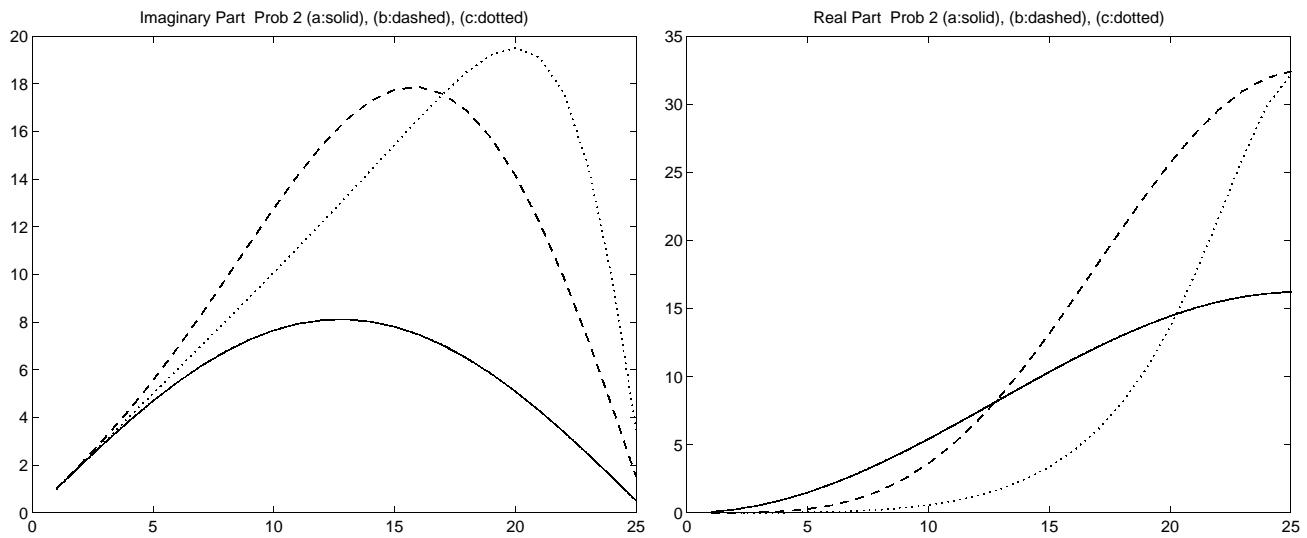
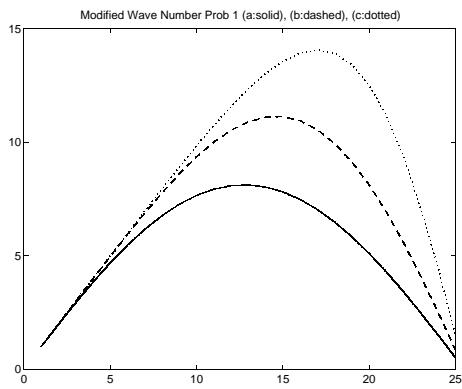
k_p1 = (1.-cos(k*dx) + i*sin(k*dx))/dx;
k_p2 =(3. - 4.*cos(k*dx) - i*sin(k*dx)) + ...
        (cos(2.*k*dx) - i*sin(2.*k*dx))/2.0/dx;
k_p3 =(2.-2.*cos(k*dx) + 3.*i*sin(k*dx))/dx./(2.+cos(k*dx) - i*sin(k*dx));

figure(2);
clf;
plot(k,imag(k_p1),'w',k,imag(k_p2),'w--',k,imag(k_p3),'w:');
title('Imaginary Part Prob 2 (a:solid), (b:dashed), (c:dotted)');

figure(3);
clf;
plot(k,real(k_p1),'w',k,real(k_p2),'w--',k,real(k_p3),'w:');
title('Real Part Prob 2 (a:solid), (b:dashed), (c:dotted)');

end

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- (a) For problem 1 the results should all be pure imaginary. Plot the imaginary part (a real number) against  $k$ .
- (b) For problem 2 the results should all be complex. Plot the imaginary part (a real number) against  $k$ . Then plot the real part against  $k$ .